Applied-Numerical Qual - August 2022

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Numerical Problem 1

Part 1

Proof. We have that

$$\begin{aligned}
a_h(v_h, v_h) &= \int_\Omega \mu |\nabla v_h|^2 + \int_\Omega v_h \beta \cdot \nabla v_h + \alpha h \int_\Omega |\nabla v_h|^2 \\
&= (\mu + \alpha h) \|\nabla v_h\|^2_{L^2(\Omega)} + \int_\Omega \frac{\beta}{2} \cdot \nabla (v_h^2) \\
&= (\mu + \alpha h) \|\nabla v_h\|^2_{L^2(\Omega)} + \int_{\partial\Omega} v_h^2 \beta \cdot \nu - \int_\Omega v_h^2 \operatorname{div}(\beta) \\
&= (\mu + \alpha h) \|\nabla v_h\|^2_{L^2(\Omega)} \\
&\geq (\mu_0 + \alpha h) \|\nabla v_h\|^2_{L^2(\Omega)} \\
&= \mu_h \|\nabla v_h\|^2_{L^2(\Omega)},
\end{aligned}$$

where we have used multidimensional integration by parts on the third line and the facts that $v_h \in H_0^1(\Omega)$ and $\operatorname{div}(\beta) = 0$.

$F(\cdot)$ is continuous by the Cauchy-Schwarz inequality, and since $\|\nabla v_h\|^2_{L^2(\Omega)}$ is an (equivalent) norm on $H_0^1(\Omega)$, $a_h(\cdot, \cdot)$ is coercive. Clearly $a_h(\cdot, \cdot)$ is bilinear, so it remains to show continuity. We have that

$$\begin{aligned}
|a_h(v_h, w_h)| &\leq (\mu + \alpha h) \int_\Omega |\nabla v_h| \, |\nabla w_h| + \int_\Omega |\beta| \, |\nabla v_h| \, |w_h| \\
&\leq (\mu + \alpha h) \|\nabla v_h\|_{L^2(\Omega)} \|\nabla w_h\|_{L^2(\Omega)} + \beta_1 \|\nabla v_h\|_{L^2(\Omega)} \|w_h\|_{L^2(\Omega)} \\
&\leq \|\nabla v_h\|_{L^2(\Omega)} (\max\{\mu + \alpha h, \beta_1\}) \|w_h\|_{H^1(\Omega)} \\
&\leq (\max\{\mu + \alpha h, \beta_1\}) \|v_h\|_{H^1(\Omega)} \|w_h\|_{H^1(\Omega)}.
\end{aligned}$$

Thus by the Lax-Milgram theorem, $a_h(u_h, v_h) = F(v_h)$ for all $v_h \in \mathbb{V}_h$ has one and only one solution. □

Part 2

Proof. We have that

$$\begin{aligned}
a_h(v_h, v_h - u_h) - F(v_h - u_h) &= a_h(v_h, v_h - u_h) - a_h(u_h, v_h - u_h) \\
&= a_h(v_h - u_h, v_h - u_h) \geq \mu_h \|\nabla (v_h - u_h)\|^2_{L^2(\Omega)}
\end{aligned}$$

by Part 1.

We have that

$$\|\nabla (u - u_h)\|_{L^2(\Omega)} \leq \|\nabla (v_h - u_h)\|_{L^2(\Omega)} + \|\nabla (v_h - u)\|_{L^2(\Omega)}.$$

We now focus on the $\|\nabla (v_h - u_h)\|_{L^2(\Omega)}$ term. Using what we've just shown, we have that

$$\|\nabla (v_h - u_h)\|_{L^2(\Omega)} \leq \frac{1}{\mu_h} \frac{|a_h(v_h, v_h - u_h) - F(v_h - u_h)|}{\|\nabla (v_h - u_h)\|_{L^2(\Omega)}} \leq \frac{1}{\mu_h} \sup_{w_h \in \mathbb{V}_h} \frac{|a_h(v_h, w_h) - F(w_h)|}{\|\nabla w_h\|_{L^2(\Omega)}}.$$

Plugging this back in yields

$$\begin{aligned}||\nabla(u-u_h)||_{L^2(\Omega)} &\le \frac{1}{\mu}\sup_{w_h\in\mathbb{V}_h}\frac{|a_h(v_h,w_h)-F(w_h)|}{||\nabla w_h||_{L^2(\Omega)}} + ||\nabla(v_h-u)||_{L^2(\Omega)} \\ &\le \frac{1}{\mu}\sup_{w_h\in\mathbb{V}_h}\frac{|a_h(v_h,w_h)-F(w_h)|}{||\nabla w_h||_{L^2(\Omega)}} + \left(1+\frac{M}{\mu_h}\right)||\nabla(v_h-u)||_{L^2(\Omega)},\end{aligned}$$

whence taking the infimum over $v_h \in \mathbb{V}_h$ yields the desired result. □

Part 3

Proof. Using the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}\sup_{w_h\in\mathbb{V}_h}\frac{|a_h(v_h,w_h)-a(v_h,w_h)|}{||\nabla w_h||_{L^2(\Omega)}} &\le \sup_{w_h\in\mathbb{V}_h}\frac{\alpha h\int_\Omega|\nabla v_h|\,|\nabla w_h|}{||\nabla w_h||_{L^2(\Omega)}} \\ &\le \sup_{w_h\in\mathbb{V}_h}\frac{\alpha h\,||\nabla v_h||_{L^2(\Omega)}\,||\nabla w_h||_{L^2(\Omega)}}{||\nabla w_h||_{L^2(\Omega)}} \\ &= \alpha h\,||\nabla v_h||_{L^2(\Omega)}.\end{aligned}$$

□

Part 4

Proof. Putting Parts 2 and 3 together, we have that

$$\begin{aligned}||\nabla(u-u_h)||_{L^2(\Omega)} &\le \frac{1}{\mu_h}\inf_{v_h\in\mathbb{V}_h}\Big(\alpha h\,||\nabla v_h||_{L^2(\Omega)} + (\mu_h+M)\,||\nabla(v_h-u)||_{L^2(\Omega)}\Big) \\ &\le \frac{\alpha h}{\mu_h}||\nabla u||_{L^2(\Omega)} + \frac{\mu_h+M+\alpha h}{\mu_h}\inf_{v_h\in\mathbb{V}_h}||\nabla(v_h-u)||_{L^2(\Omega)} \\ &\le \frac{\alpha h}{\mu_h}||u||_{H^2(\Omega)} + \frac{\mu_h+M+\alpha h}{\mu_h}\inf_{v_h\in\mathbb{V}_h}||\nabla(v_h-u)||_{L^2(\Omega)}.\end{aligned}$$

We now focus on the $\inf_{v_h\in\mathbb{V}_h}||\nabla(v_h-u)||_{L^2(\Omega)}$ term. Let $I_h u$ be the nodal Lagrange interpolant of u, then we have

$$\begin{aligned}\inf_{v_h\in\mathbb{V}_h}||\nabla(v_h-u)||_{L^2(\Omega)} \le ||\nabla(u-I_h u)||_{L^2(\Omega)} &= \sum_{T\in\mathcal{T}_h}||\nabla(u-I_h u)||_{L^2(T)} \\ &\le \sum_{T\in\mathcal{T}_h} C\,||\hat{u}-p||_{H^2(\hat{T})}\end{aligned}$$

for any $p \in \mathbb{P}^1$ (this inequality comes after transferring to the reference triangle, and since I_h disappears on $p \in \mathbb{P}^1$, we can add zero in the norm $p - I_h p$ and bound by the operator norm of I_h after collecting similar terms). Taking the infimum over $p \in \mathbb{P}^1$ and using the Bramble-Hilbert lemma yields

$$\begin{aligned}\inf_{v\in\mathbb{V}_h}||\nabla(v_h-u)||_{L^2(\Omega)} \le C\sum_{T\in\mathcal{T}_h}|\hat{u}|_{H^2(\hat{T})} &\le C\sum_{T\in\mathcal{T}_h}h_k|u|_{H^2(T)} \\ &\le C\sum_{T\in\mathcal{T}_h}h|u|_{H^2(T)} = Ch|u|_{H^2(\Omega)} \le Ch\,||u||_{H^2(\Omega)}.\end{aligned}$$

Plugging this back in, we have

$$||\nabla(u-u_h)||_{L^2(\Omega)} \le \frac{\alpha h}{\mu_h}||u||_{H^2(\Omega)} + C\frac{\mu_h+M+\alpha h}{\mu_h}h\,||u||_{H^2(\Omega)}$$

For $h \le 1$, we can bound the right-hand side by

$$\left((C+1)\frac{\alpha h}{\mu_h} + Ch\left(1+\frac{M}{\mu_h}\right)\right)||u||_{H^2(\Omega)} \le (C+1)\left(1+\frac{M+\alpha}{\mu_h}\right)||u||_{H^2(\Omega)}\,h,$$

which gives us the desired inequality. □

Numerical Problem 2

Proof. We have that

$$||v - \pi_h v||_{L^2(\Omega)} = \sum_{T\in\mathcal{T}_h} ||v - \pi_h v||_{L^2(T)} \leq C \sum_{T\in\mathcal{T}_h} \left(h_T \,||\hat{v} - \widehat{\pi_h}\hat{v}||_{L^2(\hat{T})} + ||\nabla(\hat{v} - \widehat{\pi_h}\hat{v})||_{L^2(\hat{T})} \right).$$

We remark that $\widehat{\pi_h} = \pi_h$, and since $p - \pi_h p$ disappears on $p \in \mathbb{P}^0$, for $T \in \mathcal{T}_h$ we then have

$$\begin{aligned}
&h_T \,||\hat{v} - \widehat{\pi_h}\hat{v}||_{L^2(\hat{T})} + ||\nabla(\hat{v} - \widehat{\pi_h}\hat{v})||_{L^2(\hat{T})} \\
&= h_T \,||\hat{v} - \pi_h\hat{v} - p + \pi_h p||_{L^2(\hat{T})} + ||\nabla(\hat{v} - \pi_h\hat{v} - p + \pi_h p)||_{L^2(\hat{T})} \\
&\leq h_T \,||\hat{v} - p||_{L^2(\hat{T})} + h_T \,||\pi_h(\hat{v} - p)||_{L^2(\hat{T})} + ||\nabla(\hat{v} - p)||_{L^2(\hat{T})} + ||\nabla\pi_h(\hat{v} - p)||_{L^2(\hat{T})} \\
&= h_T \,||\hat{v} - p||_{L^2(\hat{T})} + h_T \,||\pi_h(\hat{v} - p)||_{L^2(\hat{T})} + ||\nabla(\hat{v} - p)||_{L^2(\hat{T})} \\
&\leq (h_T + 1)\,||\hat{v} - p||_{H^2(\hat{T})} + h_T \,||\pi_h(\hat{v} - p)||_{L^2(\hat{T})},
\end{aligned}$$

since $\nabla\pi_h = 0$. We have that

$$||\pi_h(\hat{v} - p)||_{L^2(\hat{T})} = \left(\int_{\hat{T}} \left(\frac{1}{|\hat{T}|} \int_{\hat{T}} (\hat{v} - p)\,dx \right)^2 dx \right)^{1/2} \leq C\,||\hat{v} - p||_{L^\infty(\hat{T})} \leq C\,||\hat{v} - p||_{H^2(\hat{T})}$$

via a Sobolev embedding *(should the inner integral be over T or $\hat{T}$?)*. Thus,

$$\begin{aligned}
&h_T \,||\hat{v} - \widehat{\pi_h}\hat{v}||_{L^2(\hat{T})} + ||\nabla(\hat{v} - \widehat{\pi_h}\hat{v})||_{L^2(\hat{T})} \\
&\leq (h_T + 1 + C)\,||\hat{v} - p||_{H^2(\hat{T})} \\
&\leq C\,||\hat{v} - p||_{H^2(\hat{T})}
\end{aligned}$$

for h_T small (we reuse C as a generic constant). Furthermore,

$$||v - \pi_h v||_{L^2(\Omega)} \leq C \sum_{T\in\mathcal{T}_h} ||\hat{v} - p||_{H^2(\hat{T})};$$

after taking the infimum over $p \in \mathbb{P}^0$ and using the Bramble-Hilbert lemma, we then have

$$||v - \pi_h v||_{L^2(\Omega)} \leq C \sum_{T\in\mathcal{T}_h} |\hat{v}|_{H^1(\hat{T})}.$$

Finally, since $\hat{v} = v(B\hat{x} + b)$, we have $\nabla\hat{v} = \nabla v|B| \leq h_T \nabla v$, whence

$$||v - \pi_h v||_{L^2(\Omega)} \leq C \sum_{T\in\mathcal{T}_h} |\hat{v}|_{H^1(\hat{T})} \leq C \sum_{T\in\mathcal{T}_h} h_T\,|v|_{H^1(T)} \leq Ch \sum_{T\in\mathcal{T}_h} |v|_{H^1(T)} = Ch\,|v|_{H^1(\Omega)},$$

where C does not depend on h if h is small.

□

Numerical Problem 3

Proof. We first test with u_h^n, yielding

$$\frac{1}{\delta t}(u_h^{n+1} - u_h^n, u_h^n) + ||\nabla u_h^n||^2_{L^2(\Omega)} = 0,$$

and after using the hint, we have that

$$\frac{1}{2\delta t}\,||u_h^{n+1}||^2_{L^2(\Omega)} - \frac{1}{2\delta t}\,||u_h^n||^2_{L^2(\Omega)} + ||\nabla u_h^n||^2_{L^2(\Omega)} = \frac{1}{2\delta t}\,||u_h^{n+1} - u_h^n||^2_{L^2(\Omega)}. \tag{1}$$

We now need to estimate the right-hand side, so we test with $u_h^{n+1} - u_h^n$, yielding

$$\frac{1}{\delta t} \left|\left|u_h^{n+1} - u_h^n\right|\right|_{L^2(\Omega)}^2 = -(\nabla u_h^n, \nabla(u_h^{n+1} - u_h^n)),$$

whence

$$\frac{1}{\delta t} \left|\left|u_h^{n+1} - u_h^n\right|\right|_{L^2(\Omega)}^2 \leq ||\nabla u_h^n||_{L^2(\Omega)} \left|\left|\nabla(u_h^{n+1} - u_h^n)\right|\right|_{L^2(\Omega)},$$

after bounding by the absolute value and using Cauchy-Schwarz. Using the hint and continuing on, we have that

$$\frac{1}{\delta t} \left|\left|u_h^{n+1} - u_h^n\right|\right|_{L^2(\Omega)}^2 \leq \frac{C}{h} ||\nabla u_h^n||_{L^2(\Omega)} \left|\left|u_h^{n+1} - u_h^n\right|\right|_{L^2(\Omega)},$$

whence

$$\left|\left|u_h^{n+1} - u_h^n\right|\right|_{L^2(\Omega)} \leq \frac{C\delta t}{h} ||\nabla u_h^n||_{L^2(\Omega)}.$$

Squaring the above inequality, using it in (1), and summing over $n = 0, \cdots, N$ yields the desired result. $\square$